Convergence of a Finite Volume Scheme for the One Dimensional Vlasov-Poisson System

Francis Filbet

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Convergence of a Finite Volume scheme for the One Dimensional Vlasov-Poisson system

Francis Filbet

N° 3860

Janvier 2000

THÈME 4

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--3860--FR+ENG

INRIA LORRAINE

Convergence of a Finite Volume scheme for the One Dimensional Vlasov-Poisson system

Francis Filbet *

Thème 4 — Simulation et optimisation
de systèmes complexes
Projet Numath

Rapport de recherche n° 3860 — Janvier 2000 — 25 pages

Abstract: We propose a finite volume scheme to discretize the one-dimensional Vlasov-Poisson system, we prove that, if the initial data is positive, bounded, continuous, and has its first moment bounded, then the numerical approximation converges to the weak solution of the system for the weak topology of L^∞. Moreover, if the initial data belongs to BV, the convergence is strong in $C^0(0,T;L^1_{loc})$. To prove the convergence of the discrete electric field, we obtain an estimation in $W^{1,\infty}(\Omega_T)$.

Key-words: Finite Volume Schemes, Vlasov-Poisson system, Plasma Physics

* IECN/Inria-Lorraine, BP 239, 54506 Vandoeuvre lès Nancy, France; e-mail: filbet@loria.fr

Unité de recherche INRIA Lorraine
LORIA, Technopôle de Nancy-Brabois, Campus scientifique,
615, rue du Jardin Botanique, BP 101, 54602 Villers-Lès-Nancy (France)
Téléphone : +33 3 83 59 30 00 — Télécopie : +33 3 83 27 83 19

Convergence d'un schéma volumes finis pour Vlasov-Poisson 1D. Rapport de recherche INRIA

Résumé : On propose un schéma de type volumes finis pour la discrétisation du système de Vlasov-Poisson, on montre que si la donnée initiale est bornée, continue et a ses deux premiers moments bornés, alors l'approximation numérique ainsi construite converge pour la topologie faible de L^∞ vers la solution faible du système. De plus, si la donnée initiale appartient à BV, la convergence est forte dans $C^0([0,T];L^1_{loc})$.

Mots-clés : Schémas Volumes Finis, système de Vlasov-Poisson, Physique des plasmas

Introduction

The Vlasov-Poisson system is a model for the motion of a collisionless plasma of electrons in a uniform background of ions and describes the evolution of the distribution function of the electrons (solution of the Vlasov equation) under the effects of the transport and self consistent electric field (solution of the Poisson equation). The coupling between both equations gives a non linear problem.

The numerical resolution of the Vlasov equation is most of the time performed using particle methods (PIC), which consist in approximating the plasma by a finite number of particles. The trajectories of these particles are computed with the characteristic curves of the Vlasov equation. The interactions with self-consistent and external fields are computed by a numerical method using a mesh of the physical space (see, e.g. [2],[5]). This method enables to get satisfying results with a few number of particles.

Methods relying on a discretization of the phase space have also been proposed [10, 14] and seem to be more efficient in some cases, for example, when particles in the tail of the distribution play an important physical role, or when the numerical noise due to the finite number of particles becomes too important. Among them, the semi-Lagrangian method consists in computing directly the distribution function on a grid of the phase space. This computation is done by following the characteristic curves at each time step and interpolating the value at the feet of the characteristics by a cubic spline method [15].

This interpolation method works well for simple geometries of the physical space but does not seem to be well suited for more complex geometries.

To remedy to this problem a possible approach is to use the finite volume method which is known to be a robust and computationally cheap method for the discretization of conservation laws (see e.g. Eymard-Gallouet-Herbin [9] and the references therein). Finite volume schemes have already been implemented to approximate the solution of the Vlasov equation coupled with the Poisson equation [4, 3, 12], or with the Maxwell system [8] and the purpose of this work is to prove the convergence of a finite volume scheme for the simplest model problem in plasma physics, namely the one dimensional Vlasov-Poisson system with periodic boundary conditions (with respect to the space variable).

Before describing precisely the problem considered here, let us mention related papers where the convergence of a numerical scheme for the Vlasov-Poisson system is investigated. In [5] P.-A. Raviart and G.-H. Cottet present a precise mathematical analysis of the particle method for solving the one dimensional Vlasov-Poisson system. J. Schaeffer proves the convergence of a finite difference scheme for the Vlasov-Poisson-Fokker-Planck system [13], but he discretizes the transport part by a characteristic method, and he assumes the initial data is three times continuously differentiable. In fact, to the best of our knowledge, no convergence results seem to be available in the litterature for the numerical resolution of the Vlasov equation by an eulerian method.

We now recall the Vlasov-Poisson system: Let us set $\Omega = (0, L)$ and $\Omega_T = (0,T) \times (0, L)$, denoting by $f(t,x,v)$ the distribution function of electrons in the phase space (with mass normalized to one and the charge to plus one), and by $E(t,x)$ the self consistent electric field, the Vlasov-Poisson system reads

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + E(t,x)\frac{\partial f}{\partial v} = 0, \quad (t,x,v) \in (0,T) \times (0,L) \times \mathbb{R}, \tag{1}$$

$$\frac{\partial E}{\partial x}(t,x) = \int_{\mathbb{R}} f(t,x,v)dv \, - \, 1, \quad (t,x) \in (0,T) \times (0,L), \tag{2}$$

with a positive initial data,

$$f(0,x,v) = f_0(x,v), \qquad (x,v) \in (0,L) \times \mathbb{R}. \tag{3}$$

We impose periodic boundary conditions in x,

$$f(t,0,v) = f(t,L,v) \qquad (t,v) \in [0,T] \times \mathbb{R}, \tag{4}$$

together with the global neutrality of the plasma,

$$\frac{1}{L}\int_0^L \int_{\mathbb{R}} f(t,x,v)dvdx = 1 \qquad t \in [0,T]. \tag{5}$$

In order to completely determine the electric field $E(t,x)$, we add a zero-average electrostatic condition

$$\int_0^L E(t,x)dx = 0 \quad t \in [0,T], \tag{6}$$

which amounts to assume the electric potential is L-periodic.
We first present an up-wind finite volume scheme computing the fluxes on the boundary of each cell of the mesh, we obtain the scheme (14), and we approximate the electric field using the Green kernel.
From a L^∞ estimate on the first moment of f_h, we obtain a bound on the discrete electric field in $W^{1,\infty}$. We next give a weak BV inequality which will be useful for the convergence of the approximation to the weak solution of the Vlasov-Poisson system. We also prove that if the initial data belongs to BV, the approximation remains bounded in BV. From theese estimates,we prove

$$\begin{aligned} f_h(t,x,v) \quad &\rightharpoonup \quad f(t,x,v) \text{ in } L^\infty(Q_T) \text{ weak} - \star, \quad \text{as } h \to 0, \\ E_h(t,x) \quad &\to \quad E(t,x) \text{ in } C^0(\overline{\Omega}_T), \quad \text{as } h \to 0, \end{aligned}$$

where (f,E) is the unique solution to the Vlasov-Poisson system.
Moreover, if f_0 belongs to $BV(Q)$, then the convergence of f_h is strong in $C^0([0,T];L^1_{loc}(Q))$.

1 Regularity and discretization of the Vlasov equation solution

There are quite a number of articles adressing the existence problem in high dimension; see the survey papers of Batt [1] up to 1984 and DiPerna-Lions [7] for more recent results. Majda and Zheng prove the existence of a solution with a measure as initial data [16] for the one dimensional case with periodic boundary conditions in x. We mention in particular that Cooper and Klimas in [6] proved the global existence and uniqueness of a continuous solution $f(t,x,v)$ with $E(t,x)$ having a bounded derivative $\frac{\partial E}{\partial x}$, if the initial data $f_0(x,v)$ is continuous and its first moment is finite, in other words, there exists a positive function $R(v)$ which is decreasing in $|v|$ such that

$$f_0(x,v) \leq R(v), \text{ and } \int_0^L \int_{\mathbb{R}} |v| \, R(v)dvdx < +\infty.$$

On the other way the result of DiPerna and Lions [7], utilizing "velocity averaging" implies the existence of a renormalized solutions if $f_0(x,v)$ is assumed to satisfy the weak condition $f_0\, log^+ f_0 \in L^1((0,L)\times\mathbb{R})$.
In this paper, we will assume the initial condition is continuous and belongs to $L^\infty(Q)\cap L^1(Q)$, where $Q=(0,L)\times\mathbb{R}$, and for simplicity we will consider

$$R(v)=\frac{1}{(1+|v|)^\lambda},\quad \text{with } \lambda>2,$$

then applying the result of Klimas and Cooper, the system (1)-(6) has a unique weak solution: the couple of functions (f,E) satisfies $f(t,x,v)\in C^0(Q_T)$, $E(t,x)\in W^{1,\infty}(\Omega_T)$, and for all $\varphi\in C_c^\infty(Q_T)$

$$\int_{Q_T} f\,\Big(\frac{\partial\varphi}{\partial t}+v\,\frac{\partial\varphi}{\partial x}+E(t,x)\,\frac{\partial\varphi}{\partial v}\Big)dxdvdt+\int_Q f_0\,\varphi(0,x,v)dxdv\;=\;0.$$

and the electric field $E(t,x)$ is given by the Poisson equation

$$\frac{\partial E}{\partial x}(t,x)=\int_{\mathbb{R}} f(t,x,v)dv\;-\;1.$$

In order to compute a numerical approximation of the solution of the Vlasov-Poisson system, let us define a cartesian mesh of the phase space $\mathcal{M}_h$ constituted of cells, denoted by $C_{i,j}$, $i\in I=\{0,...,n_x-1\}$ where n_x is the number of sub-cells of $(0,L)$, and $j\in\mathbb{Z}$.
$\mathcal{M}_h$ is given by an increasing sequence $(x_{i-1/2})_{i\in\{0,...,n_x\}}$ of the interval $(0,L)$, and by a second increasing sequence $(v_{j-1/2})_{j\in\mathbb{Z}}$ of $\mathbb{R}$.
Let $\Delta x_i=x_{i+1/2}-x_{i-1/2}$ be the physical space set, and $\Delta v_j=v_{j+1/2}-v_{j-1/2}$, be the velocity space set, the parameter h indicates

$$h=\max_{i,j}\{\Delta x_i,\Delta v_j\}.$$

We assume the mesh is admissible,

$$\exists\,\alpha\in(0,1);\quad \forall\,h>0,\forall\,(i,j)\in I\times\mathbb{Z},\quad \alpha\,h\le\Delta x_i\le h\quad \text{and } \alpha\,h\le\Delta v_j\le h. \tag{7}$$

Finally, we obtain a cartesian mesh of the phase space constituted of control volumes

$$C_{i,j}=(x_{i-1/2},x_{i+1/2})\times(v_{j-1/2},v_{j+1/2}),\quad \text{for } i\in I \text{ and } j\in\mathbb{Z}.$$

Let Δt be the time step, and $t^n=n\,\Delta t$.
We set the discrete initial data $f^0_{i,j}=\frac{1}{|C_{i,j}|}\int_{C_{i,j}} f_0(x,v)dxdv$ or $f^0_{i,j}=f_0(x_i,v_j)$, where x_i (resp. v_j) represents the middle of $[x_{i-1/2},x_{i+1/2}]$ (resp. $[v_{j-1/2},v_{j+1/2}]$).
The finite volume method consists in integrating the Vlasov equation on each control volume of the mesh, approximating fluxes on the boundary,

$$\begin{aligned}\frac{1}{|C_{i,j}|}\int_{C_{i,j}} f(t^{n+1},x,v)dxdv \;&=\; \frac{1}{|C_{i,j}|}\int_{C_{i,j}} f(t^n,x,v)dxdv \\ &-\; \frac{1}{|C_{i,j}|}\big(\phi^n_{i+1/2,j}-\phi^n_{i-1/2,j}+\psi^n_{i,j+1/2}-\psi^n_{i,j-1/2}\big),\end{aligned} \tag{8}$$

where $\phi^n_{i+1/2,j}$ and $\psi^n_{i,j+1/2}$ denote the fluxes on the boundary of the cell $C_{i,j}$,

$$\begin{aligned}\phi^n_{i+1/2,j} &= \int_{t^n}^{t^{n+1}}\int_{v_{j-1/2}}^{v_{j+1/2}} v\, f(t, x_{i+1/2}, v) dv dt,\\ \psi^n_{i,j+1/2} &= \int_{t^n}^{t^{n+1}}\int_{x_{i-1/2}}^{x_{i+1/2}} E(t,x)\, f(t,x,v_{j+1/2}) dx dt.\end{aligned}$$

We approximate the fluxes $\phi^n_{i+1/2,j}$ and $\psi^n_{i,j+1/2}$ by the discrete fluxes $\bar{\phi}^n_{i+1/2,j}$ and $\bar{\psi}^n_{i,j+1/2}$ with a simple upwind scheme

$$\bar{\phi}^n_{i+1/2,j} = \begin{cases} \Delta t\, \Delta v_j\, v_j\, f^n_{i,j} & \text{if } v_j \geq 0 \\ \Delta t\, \Delta v_j\, v_j\, f^n_{i+1,j} & \text{if } v_j < 0 \end{cases}$$

and

$$\bar{\psi}^n_{i,j+1/2} = \begin{cases} \Delta t\, \Delta x_i\, E^n_i\, f^n_{i,j} & \text{if } E^n_i \geq 0 \\ \Delta t\, \Delta x_i\, E^n_i\, f^n_{i,j+1} & \text{if } E^n_i < 0, \end{cases}$$

where E^n_i is an approximation of the electric field on $[x_{i-1/2}, x_{i+1/2}]$ given below by computing an approximate solution of the Poisson equation. And $f^n_{i,j}$ is assumed to approximate the average of the Vlasov equation solution on the control volume $C_{i,j}$.
Thus, we obtain the discrete version of (8),

$$f^{n+1}_{i,j} = f^n_{i,j} - \frac{1}{|C_{i,j}|}\big(\bar{\phi}^n_{i+1/2,j} - \bar{\phi}^n_{i-1/2,j} + \bar{\psi}^n_{i,j+1/2} - \bar{\psi}^n_{i,j-1/2}\big). \tag{9}$$

To complete the scheme, we impose periodic boundary conditions on x, the values $f^n_{-1,j}$ and $f^n_{n_x,j}$ represent an approximation on a "virtual cell",

$$\begin{aligned} f^n_{n_x,j} &= f^n_{0,j} \quad \text{if } v_j \geq 0,\\ f^n_{-1,j} &= f^n_{n_x-1,j} \quad \text{if } v_j < 0.\end{aligned}$$

And, in order to work with a bounded domain, we will truncate at $|v| = v_h$ (v_h sufficiently large which will go to $+\infty$ as $h \to 0$), then we set $J = \{j \in \mathbb{Z};\quad |v_{j+1/2}| \leq v_h\}$, and impose

$$\bar{\psi}^n_{i,j+1/2} = 0 \quad \forall (i,j) \in I \times \mathbb{Z} \setminus J.$$

Thus, we are able to define the numerical solution approximating the solution of the Vlasov equation on $Q_T = \Omega_T \times \mathbb{R}$ by,

$$f_h(t,x,v) = \begin{cases} f^n_{i,j} & \text{if } (t,x,v) \in [t^n, t^{n+1}) \times C_{i,j}, \text{ and } (i,j) \in I \times J. \\ 0 & \text{if } |v| > v_h. \end{cases}$$

Computing zeroth and first order moments in v, we define the discrete charge and current densities, for $(t,x) \in [t^n, t^{n+1}) \times [x_{i-1/2}, x_{i+1/2})$

$$\begin{aligned}\rho_h(t,x) &= \int_{\mathbb{R}} f_h(t,x,v) dv = \sum_{j\in\mathbb{Z}} \Delta v_j\, f^n_{i,j} = \rho^n_i,\\ j_h(t,x) &= \int_{\mathbb{R}} v\, f_h(t,x,v) dv = \sum_{j\in\mathbb{Z}} \Delta v_j\, v_j\, f^n_{i,j} = j^n_i.\end{aligned}$$

To define a continuous approximation of the electric field, we set

$$\bar{\rho}_h(t,x) \;=\; (1 - \frac{t-t^n}{\Delta t})\,\rho_h(t^n,x) \;+\; \frac{t-t^n}{\Delta t}\,\rho_h(t^{n+1},x).$$

Now, we are able to explicitly solve the Poisson equation by the corresponding kernel

$$K(x,y) \;=\; \begin{cases} \dfrac{y}{L} - 1 & \text{if } x \le y \le L, \\[2ex] \dfrac{y}{L} & \text{if } 0 \le y \le x, \end{cases}$$

and give the discrete electric field E_h, which is continuous in (t,x), and piecewise linear

$$E_h(t,x) = \int_0^L K(x,y)\,(\bar{\rho}_h(t,y) \;-\; 1)dy. \tag{10}$$

For example, we may consider the approximation on $(x_{i-1/2}, x_{i+1/2})$, taking the value of the discrete electric field in the middle of the cell,

$$E_i^n = \frac{1}{\Delta x_i}\int_{x_{i-1/2}}^{x_{i+1/2}} E_h(t^n,x)dx \;=\; E_h(t^n,x_i) \;=\; \int_0^L K(x_i,y)\,(\rho_h(t^n,y) \;-\; 1)dy. \tag{11}$$

We shall now prove the following theorem of convergence for the numerical approximation,

Theorem 1 *Let $f_0(x,v)$ be positive, continuous and such that*

$$\exists\, C > 0; \quad f_0(x,v) \le C\,R(v); \textit{for} \quad (x,v) \in Q \tag{12}$$

where

$$R(v) = \frac{1}{(1+|v|)^\lambda} \qquad \textit{with } \lambda > 2.$$

Let $\mathcal{M}_h$ be a cartesian mesh of the phase space, Δt be the time step satisfying the CFL condition:

$$\exists\, \xi \in (0,1); \quad \frac{\Delta t}{\Delta x_i\,\Delta v_j}\big(\Delta v_j\,|v_j| + \Delta x_i\,|E_i^n|\big) \le 1-\xi, \quad \forall (i,j) \in I \times J, \quad \forall n. \tag{13}$$

If we consider the numerical solution given by the scheme (9), denoted by $f_h(t,x,v)$, and the discrete self-consistent field $E_h(t,x)$ given by (10), then we have

$$\begin{aligned} f_h(t,x,v) \;&\rightharpoonup\; f(t,x,v) \textit{ in } L^\infty(Q_T) \textit{ weak} - \star, \quad \textit{as } h \to 0, \\ E_h(t,x) \;&\to\; E(t,x) \textit{ in } C(\overline{\Omega}_T), \quad \textit{as } h \to 0, \end{aligned}$$

where (f,E) is the unique solution to the Vlasov-Poisson system (1)-(6).
Moreover, if f_0 belongs to $BV(Q)$, then the convergence of f_h is strong in $C^0([0,T];L^1_{loc}(Q))$.

Remark 1 *The main idea of the proof is to obtain an estimation of zeroth and first order moments to have an estimation on the derivatives of the electric field , then we use a compactness argument to prove the strong convergence . For the discrete distribution function a L^∞ estimate is sufficient to have the weak-$\star$ convergence.*
If the initial data belongs to $BV(Q)$, we are able to show that $f_h(t)$ remains bounded in $BV(Q)$, then we obtain the strong convergence..

2 A priori estimates

In this section, we will give some properties satisfied as well as by the numerical approximation and by the solution of the continuous problem. We can check that if the velocity set $\Delta v_j = \Delta v$ for all $j \in \mathbb{Z}$, the numerical scheme preserves the zeroth and first order moments. We will first prove a time decreasing property on $\int_Q \phi(f_h(t,x,v))dxdv$ for all convex function ϕ, which allows us to have a maximum principle on $f_h(t)$, we will also give a L^∞ estimate on the electric field $E_h(t)$. In Proposition 2, we will obtain an uniform bound in x on f_h in order to obtain an L^∞ estimate on first moment, and a $W^{1,\infty}$ estimate on the discrete electric field.

Proposition 1 *Let us assume there exists a convex function ϕ, such that*

$$\int_0^L \int_{\mathbb{R}} \phi(f_0(x,v))dxdv < +\infty.$$

Then, under the stability condition (13), the numerical solution is well defined and satisfies

$$\forall t \in \mathbb{R}^+, \tau > 0, \quad \int_0^L \int_{\mathbb{R}} \phi(f_h(t+\tau,x,v))dxdv \leq \int_0^L \int_{\mathbb{R}} \phi(f_h(t,x,v))dxdv,$$

Moreover

$$|E_h(t,x)| \leq \frac{3}{2} L \quad \forall (t,x) \in \mathbb{R}^+ \times (0.L).$$

Proof. Using the scheme (9), we explicitly write the value of the numerical solution at time t^{n+1}, in term of the values at time t^n,

$$\begin{aligned} f_{i,j}^{n+1} &= \big(1 - \Delta t(\frac{|v_j|}{\Delta x_i} + \frac{|E_i^n|}{\Delta v_j})\big) f_{i,j}^n + \Delta t \frac{v_j^+}{\Delta x_i} f_{i-1,j}^n + \Delta t \frac{v_j^-}{\Delta x_i} f_{i+1,j}^n \\ &+ \Delta t \frac{E_i^{n+}}{\Delta v_j} f_{i,j-1}^n + \Delta t \frac{E_i^{n-}}{\Delta v_j} f_{i,j+1}^n, \end{aligned} \tag{14}$$

where we use the well known notation $r^+ = max(r,0)$, and $r^- = max(-r,0)$. Under the stability condition (13), the discrete distribution function $f_{i,j}^{n+1}$ could be written as a convex combination of $f_{i,j}^n, f_{i-1,j}^n, f_{i+1,j}^n, f_{i,j-1}^n, f_{i,j+1}^n$, then considering an arbitrary convex function ϕ, we have

$$\begin{aligned} \phi\big(f_{i,j}^{n+1}\big) &\leq \big(1 - \Delta t(\frac{|v_j|}{\Delta x_i} + \frac{|E_i^n|}{\Delta v_j})\big)\, \phi\big(f_{i,j}^n\big) + \Delta t \frac{v_j^+}{\Delta x_i}\, \phi\big(f_{i-1,j}^n\big) + \Delta t \frac{v_j^-}{\Delta x_i}\, \phi\big(f_{i+1,j}^n\big) \\ &+ \Delta t \frac{E_i^{n+}}{\Delta v_j}\, \phi\big(f_{i,j-1}^n\big) + \Delta t \frac{E_i^{n-}}{\Delta v_j}\, \phi\big(f_{i,j+1}^n\big) \\ &= \phi\big(f_{i,j}^n\big). \end{aligned} \tag{15}$$

Then it follows for all $n \in \mathbb{N}$

$$\begin{aligned} \int_0^L \int_{\mathbb{R}} \phi(f_h(t^{n+1},x,v))dxdv &= \sum_{i,j} \Delta x_i \Delta v_j \phi\big(f_{i,j}^{n+1}\big) \leq \sum_{i,j} \Delta x_i \Delta v_j \phi\big(f_{i,j}^n\big) \\ &\leq \int_0^L \int_{\mathbb{R}} \phi(f_h(t^n,x,v))dxdv. \end{aligned}$$

Now, let us prove the discrete electric field is bounded, the argument is the same as in the continuous case: for (t,x) belonging to Ω_T,

$$\begin{aligned}|E_h(t,x)| &= \Big|\int_0^L K(x,y)\,(\bar\rho_h(t,y) - 1)dy\Big| \le \Big|\int_0^L K(x,y)\,\bar\rho_h(t,y)dy\Big| + \Big|\int_0^L K(x,y)dy\Big| \\ &\le \|K\|_\infty \int_0^L \bar\rho_h(t,y)dy + \frac{L}{2} \le L + \frac{L}{2} = \frac{3}{2}L. \qquad \square\end{aligned}$$

Remark 2 *As a consequence of Proposition 1, we consider $\phi(r) = r^-$ (resp. $\phi(r) = (r - \|f_0\|_\infty)^+$), and the initial data is positive (resp. bounded), then we obtain f_h is also positive (resp. bounded). We know the L^p norm of the Vlasov equation solution is preserved over the time, but this property does not seem to be satisfied by the numerical approximation, indeed if we take $\phi(r) = |r|^p$, we are only able to prove that the L^p norm is decreasing (this simple scheme is dissipative).*

Now, let us give an uniform bound in (t,x) on f_h, and an estimate on first moments ρ_h and j_h.

Proposition 2 *Assume that $0 \le f_0(x,v) \le R(v) = \frac{1}{(1+|v|)^\lambda}$, for some $\lambda > 2$. Then, there exists a constant C_T only depending on T, L, and f_0 such that*

$$0 \le f_h(t,x,v) \le C_T\, R_h(v), \qquad (t,x,v) \in (0,T) \times (0,L) \times \mathbb{R},$$

where $R_h(v) = \frac{1}{(1+|v_j|)^\lambda}$, for $v \in [v_{j-1/2}, v_{j+1/2})$, $j \in \mathbb{Z}$.

And for h small enough there exists $C_T > 0$,

$$0 \le \rho_h(t,x) \le C_T, \quad |j_h(t,x)| \le C_T, \quad (t,x) \in \Omega_T. \tag{16}$$

Proof. Let us note there exists $c_0 = c_0(\alpha,\lambda)$ such that

$$\frac{R_h(v_{j+\beta})}{R_h(v_j)} \le 1 + c_0\,\Delta v_j, \qquad \text{for } \beta = -1 \text{ or } 1,$$

then we set $A = (1 + \frac{3}{2}L\,c_0\Delta t)$, and can easily check that taking

$$f^0_{i,j} = \frac{1}{|C_{i,j}|}\int_{C_{i,j}} f_0(x,v)dxdv \quad \text{ or } \quad f^0_{i,j} = f_0(x_i,v_j),$$

we have $f_h(0,x,v) \le A^0\,R_h(v)$.
If we assume that $f_h(t^n,x,v) \le A^n\,R_h(v)$, using the numerical scheme (9)

$$\begin{aligned}\frac{f^{n+1}_{i,j}}{R_h(v_j)} &= \Big(1 - \Delta t\frac{\Delta v_j\,|v_j| + \Delta x_i\,|E^n_i|}{\Delta x_i \Delta v_j}\Big)\frac{f^n_{i,j}}{R_h(v_j)} + \Delta t\frac{v_j^+}{\Delta x_i}\frac{f^n_{i-1,j}}{R_h(v_j)} + \Delta t\frac{v_j^-}{\Delta x_i}\frac{f^n_{i+1,j}}{R_h(v_j)} \\ &+ \Delta t\frac{E^{n+}_i}{\Delta v_j}\frac{f^n_{i,j-1}}{R_h(v_{j-1})}\frac{R_h(v_{j-1})}{R_h(v_j)} + \Delta t\frac{E^{n-}_i}{\Delta v_j}\frac{f^n_{i,j+1}}{R_h(v_{j+1})}\frac{R_h(v_{j+1})}{R_h(v_j)}\end{aligned}$$

Under the CFL condition (13) and using the property of $R_h(v)$, we have

$$\begin{aligned}\frac{f_{i,j}^{n+1}}{R_h(v_j)} &\leq \Big(1-\Delta t(\frac{|v_j|}{\Delta x_i}+\frac{|E_i^n|}{\Delta v_j})\Big)\; A^n + \Delta t\frac{|v_j|}{\Delta x_i}\; A^n + \Delta t\frac{|E_i^n|}{\Delta v_j}\; A^n(1+c_0\,\Delta v_j)\\ &\leq A^n(1+\frac{3}{2}L\,c_0\,\Delta t) = A^{n+1}.\end{aligned}$$

We finally obtain

$$\forall (i,j)\in I\times\mathbb{Z},\quad \frac{f_{i,j}^{n+1}}{R_h(v_j)} \leq A^{n+1}.$$

For a finite time T and for all $n\in\{0,..,T/\Delta t\}$, $A^{n+1} < e^{c_0 T}$, then, as in the continuous case, there exists a majorizing function of the discrete distribution

$$f_h(t,x,v)\leq C_T\, R_h(v);\quad \text{for } (t,x,v)\in Q_T.$$

In order to prove the inequality (16), we observe

$$\begin{aligned}\int_{\mathbb{R}} R_h(v)dv &= \sum_{j\in\mathbb{Z}}\frac{\Delta v_j}{(1+|v_j|)^\lambda} \quad\leq\quad 2\sum_{j\in\mathbb{N}}\frac{h}{(1+\alpha\,[j-1]\,h)^\lambda}\\ &\leq h\,+\,\frac{1}{\alpha}\int_{\mathbb{R}}\frac{dv}{(1+|v|)^\lambda}\quad <\quad +\infty.\end{aligned}$$

Then, for h small enough, there exists a constant C_T only depending on f_0, α, T, L, such that

$$\rho_h(t,x) = \int_{\mathbb{R}} f_h(t,x,v)dv \leq C_T(h\,+\,\frac{1}{\alpha}\int_{\mathbb{R}}\frac{dv}{(1+|v|)^\lambda}) < +\infty,$$

and

$$|j_h(t,x)| \leq \int_{\mathbb{R}} |v|\, f_h(t,x,v)dv \leq C_T(h\,+\,\frac{1}{\alpha}\int_{\mathbb{R}}\frac{dv}{(1+|v|)^{\lambda-1}}) < +\infty.\quad\square$$

Estimation for the derivatives of E_h

In Proposition 1, we have already seen that E_h is bounded in L^∞, now we give an estimate on the derivatives.

Proposition 3 *Under the same assumptions of the Proposition 1, for h sufficiently small, there exists a constant C_T, which depends only on the initial data and on the domain, such that*

$$\Big|\frac{\partial E_h}{\partial x}(t,x)\Big| \,+\, \Big|\frac{\partial E_h}{\partial t}(t,x)\Big| \leq C_T,\quad (t,x)\in\Omega_T.$$

Proof. We first give an estimate of the derivative in x, which is explicitly given in the distribution sense by the Poisson equation: let $(t,x)\in\Omega_T$, then there exists $n\in\{0,..,T/\Delta t\}$ such that $t\in[t^n,t^{n+1})$

$$\begin{aligned}\Big|\frac{\partial E_h}{\partial x}(t,x)\Big| &= \Big|\bar\rho_h(t,x)\,-\,1\Big| \leq \Big|\bar\rho_h(t,x)\Big|\,+\,1\\ &\leq (1\,-\,\frac{t-t^n}{\Delta t})\,\rho_h(t^n,x)\,+\,\frac{t-t^n}{\Delta t}\,\rho_h(t^{n+1},x)\,+\,1\end{aligned}$$

By Proposition 2, it follows

$$\left|\frac{\partial E_h}{\partial x}(t,x)\right| \le C_T, \quad \forall (t,x) \in \Omega_T.$$

To obtain an estimate of $\frac{\partial E_h}{\partial t}$, we define a new approximation of the current density denoted by $\bar{j}_h(t,x)$, for $(t,x) \in [t^n, t^{n+1}) \times [x_{i-1/2}, x_{i+1/2})$

$$\bar{j}_h(t,x) = \bar{j}_i^n + (x - x_{i-1/2}) \frac{\bar{j}_{i+1}^n - \bar{j}_i^n}{\Delta x_i},$$

with

$$\bar{j}_i^n = \sum_{j\in\mathbb{Z}} \Delta v_j \,(\, v_j^+ \, f_{i,j}^n - \, v_j^- \, f_{i+1,j}^n \,),$$

and $\bar{j}_h \in L^\infty\big(0,T;W^{1,\infty}(\Omega)\big)$. Recalling that $\bar{\rho}_h$ defined previously belongs to $W^{1,\infty}\big(0,T;L^\infty(\Omega)\big)$, we notice that integrating (9) with respect to v yields, as in the continuous case,

$$\forall (t,x) \in \Omega_T, \quad \frac{\partial \bar{\rho}_h}{\partial t}(t,x) + \frac{\partial \bar{j}_h}{\partial x}(t,x) = 0.$$

Now,

$$\frac{\partial E_h}{\partial t}(t,x) \;=\; \int_0^L K(x,y)\,\frac{\partial \bar{\rho}_h}{\partial t}(t,y)dy = \int_0^L -K(x,y)\,\frac{\partial \bar{j}_h}{\partial x}(t,y)dy = -\bar{j}_h(t,x) + \frac{1}{L}\int_0^L \bar{j}_h(t,y)dy,$$

and observing that $|\bar{j}_i^n| \le |j_i^n| + |j_{i+1}^n|$, the Proposition 2 allows us to complete the proof. $\square$

Weak BV estimate for f_h:

The following lemma will be useful to obtain the convergence of (E_h, f_h) to the Vlasov equation solution.

Lemma 1 *Under the stability condition (13) on the time step and the condition on the mesh (7), assume the initial data belongs to $L^1(Q) \cap L^\infty(Q)$. Consider $R > 0$, and $T > 0$ with $h < R$ and $\Delta t < T$. Let $j_0, j_1 \in \mathbb{Z}$ and $N_T \in \mathbb{N}$ be such that $-R \in (v_{j_0-1/2}, v_{j_0+1/2})$,$R \in (v_{j_1-1/2}, v_{j_1+1/2})$, and $T \in ((N_T-1)\Delta t, N_T\,\Delta t)$. We define for all convex function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$*

$$\begin{aligned} EF_{1h} \;&=\; \Delta t \sum_{n=0}^{N_T} \sum_{i\in I} \sum_{j=j_0}^{j_1} \Delta x_i \Delta v_j \Bigg[v_j^+ \,|\phi(f_{i,j}^n) - \phi(f_{i-1,j}^n)| + v_j^- \,|\phi(f_{i,j}^n) - \phi(f_{i+1,j}^n)| \\ &\quad + E_i^{n+}\,|\phi(f_{i,j}^n) - \phi(f_{i,j-1}^n)| + \, E_i^{n-}\,|\phi(f_{i,j}^n) - \phi(f_{i,j+1}^n)| \Bigg]. \end{aligned}$$

and

$$EF_{2h} = \Delta t \sum_{n=0}^{N_T} \sum_{i\in I} \sum_{j=j_0}^{j_1} \Delta x_i \Delta v_j \Big| \phi(f_{i,j}^{n+1}) - \phi(f_{i,j}^n) \Big|.$$

Then there exists $C > 0$, only depending on T, R, f_0, α, ξ, such that

$$EF_{1h} \le C\, h^{1/2} \quad \text{and } EF_{2h} \le C\,\Delta t^{1/2}.$$

Proof. Let us begin by the first inequality, multiply the scheme (15) by $\Delta x_i\,\Delta v_j\,\phi(f_{i,j}^n)$, and sum over $i\in\{0,..,n_x-1\}$, $j\in\{j_0,..,j_1\}$, and $n\in\{0,..,N_T\}$, it follows:

$$B_1+B_2\le 0.$$

where:

$$B_1=\sum_{n,i,j}\Delta x_i\Delta v_j[\phi(f_{i,j}^{n+1})-\phi(f_{i,j}^n)]\,\phi(f_{i,j}^n).$$

$$\begin{aligned}B_2 &= \Delta t\sum_{n,i,j}\Big[\Delta v_j\,v_j^+\,[\phi(f_{i,j}^n)-\phi(f_{i-1,j}^n)]\,\phi(f_{i,j}^n)+\Delta v_j\,v_j^-\,[\phi(f_{i,j}^n)-\phi(f_{i+1,j}^n)]\,\phi(f_{i,j}^n)\\ &\quad+\Delta x_i\,E_i^{n+}\,[\phi(f_{i,j}^n)-\phi(f_{i,j-1}^n)]\,\phi(f_{i,j}^n)+\,\Delta x_i\,E_i^{n-}\,[\phi(f_{i,j}^n)-\phi(f_{i,j+1}^n)]\,\phi(f_{i,j}^n)\Big].\end{aligned}$$

Noting that

$$[\phi(f_{i,j}^{n+1})-\phi(f_{i,j}^n)]\,\phi(f_{i,j}^n)=-\frac{1}{2}[\phi(f_{i,j}^{n+1})-\phi(f_{i,j}^n)]^2-\frac{1}{2}\phi(f_{i,j}^n)^2+\frac{1}{2}\phi(f_{i,j}^{n+1})^2,$$

then

$$B_1=-\frac{1}{2}\sum_{n,i,j}\Delta x_i\Delta v_j[\phi(f_{i,j}^{n+1})-\phi(f_{i,j}^n)]^2-\frac{1}{2}\sum_{i,j}\Delta x_i\Delta v_j\phi(f_{i,j}^0)^2+\frac{1}{2}\sum_{i,j}\Delta x_i\Delta v_j\phi(f_{i,j}^{N_T+1})^2.$$

Owing to the scheme (15), we have

$$\begin{aligned}\sum_{n,i,j}\Delta x_i\Delta v_j[\phi(f_{i,j}^{n+1})-\phi(f_{i,j}^n)]^2 &\le \sum_{n,i,j}\frac{\Delta t^2}{\Delta x_i\Delta v_j}\Big[\Delta v_j\,v_j^+\,[\phi(f_{i,j}^n)-\phi(f_{i-1,j}^n)]+\Delta v_j\,v_j^-\,[\phi(f_{i,j}^n)-\phi(f_{i+1,j}^n)]\\ &\quad+\Delta x_i\,E_i^{n+}\,[\phi(f_{i,j}^n)-\phi(f_{i,j-1}^n)]+\,\Delta x_i\,E_i^{n-}\,[\phi(f_{i,j}^n)-\phi(f_{i,j+1}^n)]\Big]^2.\end{aligned}$$

Using the Cauchy Schwarz inequality, and the stability condition (13)

$$\begin{aligned}B_1 &\ge -\frac{1}{2}\Delta t(1-\xi)\sum_{n,i,j}\Big[\Delta v_j\,v_j^+\,[\phi(f_{i,j}^n)-\phi(f_{i-1,j}^n)]^2+\Delta v_j\,v_j^-\,[\phi(f_{i,j}^n)-\phi(f_{i+1,j}^n)]^2\\ &\quad+\Delta x_i\,E_i^{n+}\,[\phi(f_{i,j}^n)-\phi(f_{i,j-1}^n)]^2+\,\Delta x_i\,E_i^{n-}\,[\phi(f_{i,j}^n)-\phi(f_{i,j+1}^n)]^2\Big]\\ &\quad-\frac{1}{2}\sum_{i,j}\Delta x_i\Delta v_j\phi(f_{i,j}^0)^2.\end{aligned}$$

We now study the term B_2, which may be rewritten as

$$\begin{aligned}B_2 &= \frac{1}{2}\Delta t\sum_{n,i,j}\Big[\Delta v_j\,v_j^+\,[\phi(f_{i,j}^n)-\phi(f_{i-1,j}^n)]^2+\Delta v_j\,v_j^-\,[\phi(f_{i,j}^n)-\phi(f_{i+1,j}^n)]^2\\ &\quad+\Delta x_i\,E_i^{n+}\,[\phi(f_{i,j}^n)-\phi(f_{i,j-1}^n)]^2+\,\Delta x_i\,E_i^{n-}\,[\phi(f_{i,j}^n)-\phi(f_{i,j+1}^n)]^2\Big]\\ &\quad+\frac{1}{2}\Delta t\sum_{n,i}\Big[\Delta x_i\,E_i^{n+}\,[\phi(f_{i,j_1}^n)^2-\phi(f_{i,j_0-1}^n)^2]+\,\Delta x_i\,E_i^{n-}\,[\phi(f_{i,j_0}^n)^2-\phi(f_{i,j_1+1}^n)^2]\Big].\end{aligned}$$

Then, since $B_1 + B_2 \leq 0$ the following inequality holds

$$\begin{aligned}
\Delta t \ & \sum_{n,i,j} \Bigg[\Delta v_j\, v_j^+\, [\phi(f_{i,j}^n) - \phi(f_{i-1,j}^n)]^2 + \Delta v_j\, v_j^-\, [\phi(f_{i,j}^n) - \phi(f_{i+1,j}^n)]^2 \\
& \quad + \Delta x_i\, E_i^{n+}\, [\phi(f_{i,j}^n) - \phi(f_{i,j-1}^n)]^2 + \Delta x_i\, E_i^{n-}\, [\phi(f_{i,j}^n) - \phi(f_{i,j+1}^n)]^2 \Bigg] \\
\leq & \ \frac{1}{\xi} \sum_{i,j} \Delta x_i \Delta v_j \phi(f_{i,j}^0)^2 + \frac{\Delta t}{\xi} \sum_{n,i} \Delta x_i\, |E_i^n|\, [\phi(f_{i,j_0-1}^n)^2 + \phi(f_{i,j_1+1}^n)^2] \\
\leq & \ \frac{1}{\xi} \int_{Q_T} |\phi(f_h(0))|^2 dx dv + \frac{2}{\xi} \|\phi(f_0)\|_\infty^2 \, \|E_h\|_{L^1(\Omega_T)} = \frac{K}{\xi},
\end{aligned}$$

Remark that K does not depend on h, indeed

$$\|E_h\|_{L^1(\Omega_T)} \leq T\, \|f_0\|_{L^1(Q)}, \text{ and } \Big(\int_{Q_T} |\phi(f_h(0))|^2 dx dv \Big)^{1/2} \leq T\, \|\phi(f_0)\|_{L^2(Q)}.$$

Finally, the previous inequality and the Cauchy Schwarz inequality lead to

$$\begin{aligned}
EF_{1h} \ \leq & \ \Bigg[\Delta t \sum_{n,i,j} \Delta v_j\, v_j^+\, [\phi(f_{i,j}^n) - \phi(f_{i-1,j}^n)]^2 + \Delta v_j\, v_j^-\, [\phi(f_{i,j}^n) - \phi(f_{i+1,j}^n)]^2 \\
+ & \ \Delta x_i\, E_i^{n+}\, [\phi(f_{i,j}^n) - \phi(f_{i,j-1}^n)]^2 + \Delta x_i\, E_i^{n-}\, [\phi(f_{i,j}^n) - \phi(f_{i,j+1}^n)]^2 \Bigg]^{1/2} \\
\times & \ \Bigg[\Delta t \sum_{n,i,j} \Delta x_i^2 (\Delta v_j\, |v_j| + \Delta x_i\, |E_i^n|) \Bigg]^{1/2} \\
\leq & \ h^{1/2} \left(\frac{K}{\xi} \right)^{1/2} \left[2\, T\, L\, R\, (R + \frac{3}{2} L) \right]^{1/2}.
\end{aligned}$$

Now we prove the second estimate on EF_{2h}, using the inequality (15)

$$\begin{aligned}
EF_{2h} \ = & \ \Delta t \sum_{n,i,j} \Delta x_i \Delta v_j \Big| \phi(f_{i,j}^{n+1}) - \phi(f_{i,j}^n) \Big| \\
\leq & \ \Delta t^2 \sum_{n,i,j} \Bigg[\Delta v_j\, v_j^+\, |\phi(f_{i,j}^n) - \phi(f_{i-1,j}^n)| + \Delta v_j\, v_j^-\, |\phi(f_{i,j}^n) - \phi(f_{i+1,j}^n)| \\
& \quad + \Delta x_i\, E_i^{n+}\, |\phi(f_{i,j}^n) - \phi(f_{i,j-1}^n)| + \Delta x_i\, E_i^{n-}\, |\phi(f_{i,j}^n) - \phi(f_{i,j+1}^n)| \Bigg]
\end{aligned}$$

As in the previous case, we use the Cauchy Schwarz inequality and the stability condition (13), we also recall that the discrete electric field is uniformly bounded,

$$EF_{2h} \leq \Delta t^{1/2}\, K^{1/2} \left[2\, T\, L\, R\, \frac{1-\xi}{\xi} \right]^{1/2}. \qquad \square$$

Strong BV estimate estimate:

In this section, we will assume that the initial data $f_0(x,v)$ belongs to $BV(Q)$. In order to obtain the strong convergence in $L^1_{loc}(Q)$, we will obtain an estimation on the total variation of $f_h(t)$.

Preliminary: Since our numerical approximations are function of several variables, we generalize the definition of the total variation [11]. To simplify, we give the definition for a function with two variables (x,y).

Definition 1 *Let $g(x,y)$ be a function defined on $\mathbb{R}^2$. The total variation of g is the number given by the following limit*

$$TV_{xy}(g) = \limsup_{\varepsilon\to 0} \frac{1}{\varepsilon}\int |g(x+\varepsilon,y)-g(x,y)|dxdy + \limsup_{\varepsilon\to 0}\frac{1}{\varepsilon}\int |g(x,y+\varepsilon)-g(x,y)|dxdy.$$

We can define the total variation of a piecewise constant function g analogously by

$$TV_{xy}(g) = \sum_{i,j} |y_{j+1}-y_j|\,|g(x_{i+1},y_j)-g(x_i,y_j)| + |x_{i+1}-x_i|\,|g(x_i,y_{j+1})-g(x_i,y_j)|.$$

Proposition 4 *(Total variation estimate for the discrete distribution function f_h)*
Under the stability condition (13), and if the initial data f_0 belongs to $BV(Q)$, then there exists a constant C_T, which only depends on f_0, L, and T, such that,

$$\forall n\in\{0,...,T/\Delta t\},\quad TV_{xv}(f_h(t^n)) \leq C_T\, TV_{xv}(f_0).$$

Proof. Let us write the scheme (9) on cells i and $i+1$, making the difference between both terms, and using the fact

$$\begin{aligned} E^{n+}_{i+1} &= E^{n+}_{i+1} - E^{n+}_i + E^{n+}_i, \\ E^{n-}_{i+1} &= E^{n-}_{i+1} - E^{n-}_i + E^{n-}_i, \end{aligned}$$

we directly obtain

$$\begin{aligned} f^{n+1}_{i+1,j} - f^{n+1}_{i,j} &= \Big(1-\Delta t|\frac{v_j^+}{\Delta x_{i+1}} + \frac{v_j^-}{\Delta x_i} + \frac{|E^n_i|}{\Delta v_j}|\Big)(f^n_{i+1,j}-f^n_{i,j}) \\ &+ \Delta t\frac{v_j^+}{\Delta x_i}(f^n_{i,j}-f^n_{i-1,j}) + \Delta t\frac{v_j^-}{\Delta x_{i+1}}(f^n_{i+2,j}-f^n_{i+1,j}) \\ &+ \Delta t\frac{E^{n+}_i}{\Delta v_j}(f^n_{i+1,j-1}-f^n_{i,j-1}) + \Delta t\frac{E^{n-}_i}{\Delta v_j}(f^n_{i+1,j+1}-f^n_{i,j+1}) \\ &+ \Delta t\frac{E^{n+}_{i+1}-E^{n+}_i}{\Delta v_j}(f^n_{i+1,j}-f^n_{i+1,j-1}) + \Delta t\frac{E^{n-}_{i+1}-E^{n-}_i}{\Delta v_j}(f^n_{i+1,j+1}-f^n_{i+1,j}). \end{aligned}$$

Let us multiply by Δv_j, sum over $i \in \{0,..,n_x-1\}$ and $j \in \mathbb{Z}$,

$$
\begin{aligned}
&\sum_{i,j} \Delta v_j \,|f^{n+1}_{i+1,j} - f^{n+1}_{i,j}| \\
&\leq \sum_{i,j} \Delta v_j \left(1 - \Delta t |\frac{v_j^+}{\Delta x_{i+1}} + \frac{v_j^-}{\Delta x_i} + \frac{|E_i^n|}{\Delta v_j}|\right) |f^n_{i+1,j} - f^n_{i,j}| && (17)\\
&+ \Delta t \sum_{i,j} \Delta v_j \,\frac{v_j^+}{\Delta x_i} |f^n_{i,j} - f^n_{i-1,j}| + \Delta t \sum_{i,j} \Delta v_j \,\frac{v_j^-}{\Delta x_{i+1}} |f^n_{i+2,j} - f^n_{i+1,j}| && (18)\\
&+ \Delta t \sum_{i,j} E_i^{n+} |f^n_{i+1,j-1} - f^n_{i,j-1}| + \Delta t \sum_{i,j} E_i^{n-} |f^n_{i+1,j+1} - f^n_{i,j+1}| && (19)\\
&+ \Delta t \sum_{i,j} |E^{n+}_{i+1} - E^{n+}_i| \, |f^n_{i+1,j} - f^n_{i+1,j-1}| \\
&+ \Delta t \sum_{i,j} |E^{n-}_{i+1} - E^{n-}_i| \, |f^n_{i+1,j+1} - f^n_{i+1,j}|.
\end{aligned}
$$

We use the fact f_h is periodic in x to treat terms (17)-(19), and recall that under the stability condition (13) the sum of coefficients in front of the term $|f^n_{i+1,j} - f^n_{i,j}|$ is equal to one, finally we obtain the inequality,

$$
\begin{aligned}
\sum_{i,j} \Delta v_j \,|f^{n+1}_{i+1,j} - f^{n+1}_{i,j}| &\leq \sum_{i,j} \Delta v_j \,|f^n_{i+1,j} - f^n_{i,j}| && (20)\\
&+ \Delta t \sum_{i,j} |E^{n+}_{i+1} - E^{n+}_i| \, |f^n_{i+1,j} - f^n_{i+1,j-1}| && (21)\\
&+ \Delta t \sum_{i,j} |E^{n-}_{i+1} - E^{n-}_{i+1}| \, |f^n_{i+1,j+1} - f^n_{i+1,j}|. && (22)
\end{aligned}
$$

Now, we have to study the terms (21)-(22), which represent a total variation of f_h at time t^n in function of the velocity variable v, we recall the discrete electric field is Lipschitz continuous in x, then

$$\exists\, c_{1,T} > 0, \quad \left|E^n_{i+1} - E^n_i\right| \leq \mathrm{M}_T \,\Delta x_i,$$

where M_T is a constant which only depends on the domain, on the initial data and on the final time T.
We use the fact that the function $x \mapsto \max(x,0)$ is Lipschitz continuous, with a constant equal to 1.

$$
\begin{aligned}
\sum_{i,j} \Delta v_j \,|f^{n+1}_{i+1,j} - f^{n+1}_{i,j}| &\leq \sum_{i,j} \Delta v_j \,|f^n_{i+1,j} - f^n_{i,j}| \\
&+ \Delta t \sum_{i,j} |E^n_{i+1} - E^n_i| \left(|f^n_{i+1,j} - f^n_{i+1,j-1}| + |f^n_{i+1,j+1} - f^n_{i+1,j}|\right) \\
&\leq \sum_{i,j} \Delta v_j \,|f^n_{i+1,j} - f^n_{i,j}| \,+\, \Delta t \sum_{i,j} c_{1,T} \,\Delta x_i \,|f^n_{i,j+1} - f^n_{i,j}|.
\end{aligned}
$$

We finally obtain an estimate of the total variation in x of the discrete distribution function at time t^{n+1} in function of the total variation of the discrete distribution at time t^n.
TV_x denotes the total variation in x, in fact

$$\begin{aligned} TV_x\Big(f_h(t^{n+1})\Big) &= \sum_{i,j} \Delta v_j \, |f^{n+1}_{i+1,j} - f^{n+1}_{i,j}| \\ &\leq TV_x\Big(f_h(t^n)\Big) + c_{1,T}\,\Delta t\, TV_v\Big(f_h(t^n)\Big). \end{aligned} \tag{23}$$

By a similar argument, using the fact the mesh is admissible (7),

$$v_{j+1} - v_j \leq (\frac{1}{2} + \frac{1}{2\alpha})\,\Delta v_j,$$

we obtain an estimate of the total variation at time t^{n+1} in v,

$$\exists\, c_{2,T} > 0, \quad TV_v\Big(f_h(t^{n+1})\Big) \leq TV_v\Big(f_h(t^n)\Big) + c_{2,T}\,\Delta t\, TV_x\Big(f_h(t^n)\Big). \tag{24}$$

Thus, with inequalities (23) and (24), we express the total variation estimate at time t^{n+1} in function of the total variation at time t^n, we set $c_{3,T} = c_{1,T} + c_{2,T}$

$$TV_{xv}\Big(f_h(t^{n+1})\Big) \leq TV_{xv}\Big(f_h(t^n)\Big) + c_{3,T}\,\Delta t\, TV_{xv}\Big(f_h(t^n)\Big). \tag{25}$$

Then,

$$TV_{xv}\Big(f_h(t^n)\Big) \leq \exp(c_{3,T}T)\, TV_{xv}\Big(f_h(0)\Big) \leq exp(c_{3,T}\,T)\, TV_{xv}\Big(f_0\Big). \quad \square$$

Remark 3 *To achieve the proof of the Proposition 4, we use the fact that if f_0 belongs to $BV(Q)$, then it satisfies the following inequality*

$$\sum_{i,j} \Delta v_j \, |f^0_{i+1,j} - f^0_{i,j}| \,+\, \Delta x_i \, |f^0_{i,j+1} - f^0_{i,j}| \leq TV_{xv}(f_0).$$

3 Proof of theorem 1

We consider a sequence of a mesh of the phase space defined as in the begining of the paper satisfying the condition (7), and define a time step Δt such that the stability condition (13) is true. This sequence is denoted by $(\mathcal{M}_h)_{h>0}$.
For a given mesh, we are able to construct by the finite volume scheme (9)-(11), a unique pair (f_h, E_h). Thus, we set

$$\mathcal{A} = \Big\{ E_h \in W^{1,\infty}(\Omega_T)\,; \quad E_h \text{ given by (11) for a mesh } \mathcal{M}_h \Big\}.$$

On the one hand, in the Proposition 3 we have proved there exists a constant independant on the mesh $\mathcal{M}_h$, such that

$$\forall E_h \in \mathcal{A}, \quad \|E_h\|_\infty \,+\, \|\frac{\partial E_h}{\partial t}\|_\infty \,+\, \|\frac{\partial E_h}{\partial x}\|_\infty \leq C_T.$$

On the other hand, using the fact the injection from $W^{1,\infty}(\Omega_T)$ to $C^0(\overline{\Omega}_T)$ is compact, then there exists a subsequence of $(E_h)_{h>0}$, and a function E belonging to $C^0(\overline{\Omega}_T)$ such that

$$E_h(t,x) \rightharpoonup E(t,x) \quad \text{in } L^\infty(\Omega_T) \text{ weak-}\star \quad \text{as} \quad h \to 0,$$

$$E_h(t,x) \to E(t,x) \quad \text{in } C^0(\overline{\Omega}_T) \text{ strong} \quad \text{as} \quad h \to 0.$$

Moreover, we also know by the Proposition 1, that the discrete distribution function f_h is bounded in $L^\infty(Q_T)$, then there exists a subsequence, and a function $f \in L^\infty(Q_T)$ such that,

$$f_h(t,x,v) \rightharpoonup f(t,x,v) \quad \text{in } L^\infty(Q_T) \text{ weak-}\star \quad \text{as} \quad h \to 0.$$

the discrete charge ρ_h is bounded in $L^\infty(\Omega_T)$, then up to the extraction of a subsequence, we also have

$$\rho_h(t,x) \rightharpoonup \rho(t,x) \quad \text{in } L^\infty(\Omega_T) \text{ weak-}\star \quad \text{as} \quad h \to 0.$$

Let us prove that the limit $\rho(t,x)$ is equal to $\int_{\mathbb{R}} f(t,x,v)dv$: consider $\psi(t,x) \in L^1(\Omega_T)$, then we have

$$\begin{aligned}\int_0^T \int_0^L \Big(\rho_h - \int_{\mathbb{R}} f\,dv\Big)\psi(t,x)\,dxdt &= \int_0^T \int_0^L \int_{|v|\le r} \Big(f_h - f\Big)\psi(t,x)\,dvdxdt \\ &+ \int_0^T \int_0^L \int_{|v|>r} \Big(f_h - f\Big)\psi(t,x)\,dvdxdt.\end{aligned}$$

Since $f_h \rightharpoonup f$ in L^∞ weak-$\star$, the first term of the right hand side goes to zero for every fixed r. Moreover, from the second estimate of the Proposition 1, we have

$$\int_{|v|>r} |f_h - f|dv \le \int_{|v|>r} (|f_h| + |f|)dv \le 2\,C_T\,\Big(h + \int_{|v|>r} \frac{dv}{(1+|v|)^\lambda}\Big).$$

Then, the second term can be small by choosing r large enough uniformly with respect to h, thus ρ_h converges to $\int_{\mathbb{R}} f\,dv$ in $L^\infty(\Omega_T)$ weak-$\star$.
Moreover, if we assume the initial data belongs to $BV(Q)$, then we construct a new approximation of the distribution function, continuous in time, denoted by $\bar{f}_h$ (it is easy to prove that f_h and $\bar{f}_h$ converge to the same limit), and we set

$$\mathcal{B} = \Big\{\bar{f}_h \in C(0,T;L^1_{loc}(Q))\,; \quad \bar{f}_h \text{ given by (9) for a mesh } \mathcal{M}_h\Big\},$$

and

$$\mathcal{B}(t) = \Big\{\bar{f}_h(t) \in L^1_{loc}(Q)\,; \quad \bar{f}_h \in \mathcal{B}\Big\}.$$

A consequence of the Helly compactness theorem and the total variation estimate of the discrete distribution function infer that $\mathcal{B}(t) \subset BV(Q)$, then $\mathcal{B}(t)$ is relatively compact in $L^1_{loc}(Q)$. Furthermore using the continuity of $\bar{f}_h$ we can prove that $\mathcal{B}$ is uniformly equicontinuous:

$$\forall \varepsilon > 0, \quad \exists \eta > 0, \quad \|f_h(t_1) - f_h(t_2)\|_{L^1_{loc}} \le \varepsilon, \quad f_h \in \mathcal{B}, \quad 0 \le t_1 \le t_2 \le T, |t_1 - t_2| \le \eta.$$

Then, applying the Ascoli theorem we prove that $\bar{f}_h$ strongly converges to f in $C^0(0,T;L^1_{loc})$.

Convergence to the weak solution of the Vlasov equation

Let $\varphi \in C_c^\infty(Q_T)$, let $R > 0$ and $j_0, j_1 \in \mathbb{Z}$ be such that

$$supp\Big(\varphi(t,x,.)\Big) \subset [-R, R],$$

and

$$-R \in (v_{j_0-1/2}, v_{j_0+1/2}), \qquad \text{and} \qquad R \in (v_{j_1-1/2}, v_{j_1+1/2}).$$

We multiply the finite volume scheme (9) by $\frac{1}{\Delta t\,\Delta x_i\,\Delta v_j}\int_{t^n}^{t^{n+1}}\int_{C_{i,j}}\varphi(t,x,v)dxdvdt$, sum over $i \in \{0,..,nx-1\}$, $j \in \{j_0,..,j_1\}$, and $n \in \{0,..,N_T = \frac{T}{\Delta t}\}$

$$E_1 + E_2 = 0.$$

with

$$E_1 = \sum_{n,i,j}(f_{i,j}^{n+1} - f_{i,j}^n)\frac{1}{\Delta t}\int_{t^n}^{t^{n+1}}\int_{C_{i,j}}\varphi(t,x,v)dxdvdt.$$

and

$$\begin{aligned} E_2 &= \sum_{n,i,j}\Bigg[\Delta v_j\, v_j^+\,(f_{i,j}^n - f_{i-1,j}^n) + \Delta v_j\, v_j^-\,(f_{i,j}^n - f_{i+1,j}^n) + \Delta x_i\, E_i^{n+}\,(f_{i,j}^n - f_{i,j-1}^n) \\ &\quad + \Delta x_i\, E_i^{n-}\,(f_{i,j}^n - f_{i,j+1}^n)\Bigg]\frac{1}{\Delta x_i\,\Delta v_j}\int_{t^n}^{t^{n+1}}\int_{C_{i,j}}\varphi(t,x,v)dxdvdt. \end{aligned}$$

Moreover, we denote by

$$E_{1,0} = \int_{Q_T} f_h(t,x,v)\frac{\partial\varphi}{\partial t}(t,x,v)dtdxdv + \int_Q f_0(x,v)\varphi(0,x,v)dxdv.$$

and

$$E_{2,0} = \int_{Q_T} f_h(t,x,v)\Big(v\,\frac{\partial\varphi}{\partial x}(t,x,v) \;+\; E_h(t,x)\,\frac{\partial\varphi}{\partial v}(t,x,v)\Big)dxdvdt.$$

We will compare E_1 with $E_{1,0}$, and E_2 with $E_{2,0}$ to establish that $E_{1,0} + E_{2,0}$ goes to zero as $h \to 0$.

Comparison between E_1 and $E_{1,0}$

Let us remark that $E_{1,0}$ can be rewritten as

$$E_{1,0} = \sum_{n,i,j} f_{i,j}^n \int_{C_{i,j}}\Big(\varphi(t^{n+1},x,v) - \varphi(t^n,x,v)\Big)dxdv + \int_Q f_0(x,v)\varphi(0,x,v)dxdv.$$

By a discrete integration by part, it follows

$$\begin{aligned} E_{1,0} &= -\sum_{n,i,j}\Big(f_{i,j}^{n+1} - f_{i,j}^n\Big)\int_{C_{i,j}}\varphi(t^{n+1},x,v)dxdv \\ &\quad - \int_Q\Big(f_h(0,x,v) - f_0(x,v)\Big)\varphi(0,x,v)dxdv. \end{aligned}$$

Thus

$$
\begin{aligned}
|E_1 + E_{1,0}| \quad &\leq \quad \sum_{n,i,j} |f_{i,j}^{n+1} - f_{i,j}^n| \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} |\frac{\partial \varphi}{\partial t}(t,x,v)| dt dx dv \\
&+ \quad \int_Q |f_h(0,x,v) - f_0(x,v)| \; |\varphi(0,x,v)| dx dv.
\end{aligned}
$$

with the discrete initial data defines for example by

$$
f_h(0,x,v) = \frac{1}{|C_{i,j}|} \int_{C_{i,j}} f_0(x,v) dx dv, \quad \forall (x,v) \in C_{i,j}.
$$

Using the assumption on the initial data $f_0 \in L^1(Q) \cap L^\infty(Q)$, then

$$
\lim_{h \to 0} \int_Q |f_h(0,x,v) - f_0(x,v)| \; |\varphi(0,x,v)| dx dv = 0.
$$

Moreover, from the inequality on the term EF_{2h} (17) in Lemma 1 (taking $\phi(r) = r$ for $r \in \mathbb{R}^+$), we have :

$$
\sum_{n,i,j} |f_{i,j}^{n+1} - f_{i,j}^n| \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} |\frac{\partial \varphi}{\partial t}(t,x,v)| dt dx dv \leq C \|\varphi_t\|_\infty \, \Delta t^{1/2}.
$$

Then

$$
|E_1 + E_{1,0}| \to 0, \qquad \text{as } h \to 0.
$$

Comparison between E_2 and $E_{2,0}$

We first introduce the notation

$$
\begin{aligned}
E_{2,1} \quad = \quad \sum_{n,i,j} \Bigg[& v_j^+ \, (f_{i,j}^n - f_{i-1,j}^n) \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{i-1/2}, v) dv dt \\
& + v_j^- \, (f_{i,j}^n - f_{i+1,j}^n) \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{i+1/2}, v) dv dt \\
& + E_i^{n+} \, (f_{i,j}^n - f_{i,j-1}^n) \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t, x, v_{j-1/2}) dx dt \\
& + E_i^{n-} \, (f_{i,j}^n - f_{i,j+1}^n) \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t, x, v_{j+1/2}) dx dt \Bigg].
\end{aligned}
$$

On the one hand we compare E_2 and $E_{2,1}$

$$\begin{aligned}
|E_2 - E_{2,1}| &= \Bigg| \sum_{n,i,j} \Bigg[v_j^+ \left(f_{i,j}^n - f_{i-1,j}^n\right) \left[\frac{1}{\Delta x_i} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t,x,v) - \varphi(t, x_{i-1/2}, v) dv dt \right] \\
&\quad + v_j^- \left(f_{i,j}^n - f_{i+1,j}^n\right) \left[\frac{1}{\Delta x_i} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t,x,v) - \varphi(t, x_{i+1/2}, v) dv dt \right] \\
&\quad + E_i^{n+} \left(f_{i,j}^n - f_{i,j-1}^n\right) \left[\frac{1}{\Delta v_j} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t,x,v) - \varphi(t, x, v_{j-1/2}) dx dt \right] \\
&\quad + E_i^{n-} \left(f_{i,j}^n - f_{i,j+1}^n\right) \left[\frac{1}{\Delta v_j} \int_{t^n}^{t^{n+1}} \int_{C_{i,j}} \varphi(t,x,v) - \varphi(t, x, v_{j-1/2}) dx dt \right] \Bigg] \Bigg|.
\end{aligned}$$

Using the inequality on EF_{1h} (17) in the Lemma 1 with $\phi(r) = r$, there exists $c > 0$ only depending on T, R, L, f_0, α, ξ such that the following inequality holds

$$|E_2 - E_{2,1}| \leq c \, \|\nabla\varphi\|_\infty \, h^{1/2}.$$

On the other hand we estimate $|E_{2,0} + E_{2,1}|$, we rewrite the term $E_{2,1}$ as follow (we recall that φ has a compact support)

$$\begin{aligned}
E_{2,1} &= -\sum_{n,i,j} f_{i,j}^n \Bigg[v_j \int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \varphi(t, x_{i+1/2}, v) - \varphi(t, x_{i-1/2}, v) dv dt \\
&\quad + E_i^n \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \varphi(t, x, v_{j+1/2}) - \varphi(t, x, v_{j-1/2}) dx dt \Bigg].
\end{aligned}$$

In the same way

$$\begin{aligned}
E_{2,0} &= \sum_{n,i,j} f_{i,j}^n \Bigg[\int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} v \big(\varphi(t, x_{i+1/2}, v) - \varphi(t, x_{i-1/2}, v)\big) dv dt \\
&\quad + \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} E_h(t,x) \big(\varphi(t, x, v_{j+1/2}) - \varphi(t, x, v_{j-1/2})\big) dx dt \Bigg].
\end{aligned}$$

Therefore, there exists $c > 0$ only depending on T, R, L, f_0, α,ξ

$$\begin{aligned}
|E_{2,0} + E_{2,1}| &\leq \sum_{n,i,j} f_{i,j}^n \Bigg[\int_{t^n}^{t^{n+1}} \int_{v_{j-1/2}}^{v_{j+1/2}} \big|v - v_j\big| \, \big|\varphi(t, x_{i+1/2}, v) - \varphi(t, x_{i-1/2}, v)\big| dv dt \\
&\quad + \int_{t^n}^{t^{n+1}} \int_{x_{i-1/2}}^{x_{i+1/2}} \big|E_h(t,x) - E_i^n\big| \, \big|\varphi(t, x, v_{j+1/2}) - \varphi(t, x, v_{j-1/2})\big| dx dt \Bigg] \\
&\leq c \, \|\nabla\varphi\|_\infty \sum_{n,i,j} \Delta t \, \Delta x_i \, \Delta v_j \, f_{i,j}^n \, \Big[\Delta v_j + \sup |E_h(t,x) - E_i^n| \Big] \\
&\leq c T \, \|\nabla\varphi\|_\infty \, \|f_0\|_1 \, h.
\end{aligned}$$

Finally, recalling that $E_1 + E_2 = 0$, we obtain

$$\begin{aligned}\mathcal{E}(\Delta t, h) &= \int_{Q_T} f_h\Big(\frac{\partial\varphi}{\partial t} + v\,\frac{\partial\varphi}{\partial x} \,+\, E_h(t,x)\,\frac{\partial\varphi}{\partial v}\Big)dtdxdv + \int_Q f_0(x,v)\varphi(0,x,v)dxdv \\ &= E_{1,0} + E_{2,0} \\ &= E_{1,0} + E_1 + E_{20} + E_{2,1} - E_{2,1} + E_2.\end{aligned}$$

and from the previous estimates, we proved there exists a constant C only depending on φ, f_0, L, T, α, ξ, such that

$$\begin{aligned}|E_{1,0} + E_1| &\le C\,(\|f_0 - f_h(0)\|_1 + \Delta t^{1/2}), \\ |E_{1,0} + E_1| &\le C\,h^{1/2}, \\ |E_{1,0} + E_1| &\le C\,h.\end{aligned}$$

Then $\mathcal{E}(\Delta t, h) \to 0$ as $h \to 0$.
As we know

$$f_h(t,x,v) \rightharpoonup f(t,x,v) \text{ in } L^\infty(Q_T) \text{ weak-}\star.$$

and

$$E_h(t,x) \to E(t,x) \text{ in } C^0(\overline{\Omega}_T),$$

we have shown that the pair (f, E) limit of a subsequence $(f_h, E_h)_{h>0}$ is a solution of the Vlasov equation (1). To conclude, we have to prove this couple is also solution of the Poisson equation (2).

Remark 4 *In practical calculation, we use a large but finite bound M for the velocity space. In this paper, we assume that as $h \to 0$, the support of the velocity space goes to infinity, the stability condition (13) imposes us that*

$$\exists\, \varepsilon \in (0,1), \quad v_h \simeq \frac{1}{h^\epsilon}, \text{ and } \Delta t \simeq \frac{h^2}{h^{1-\epsilon} + h} \simeq h^{1+\epsilon}.$$

Convergence to the solution of the Poisson equation:

We recall that the discrete electric field defined before is continuous in time, but for a simpler analysis let us consider a new approximation piecewise constant in time

$$\tilde{E}_h(t,x) \;=\; \int_0^L K(x,y)\,(\rho_h(t,y) \,-\, 1)dy.$$

Recalling $\frac{\partial E_h}{\partial t}$ is uniformly bounded, it is easy to prove that $\tilde{E}_h$ and E_h have the same behaviour as h goes to zero, then $\tilde{E}_h$ converges almost every where to E.
Let us prove $E(t,x)$ is solution of the Poisson equation, let $\psi(t,x)$ belong to $L^1(\Omega_T)$,

$$\int_{\Omega_T} \tilde{E}_h(t,x)\,\psi(t,x)dtdx = \int_{\Omega_T}\Big[\int_0^L K(x,y)\,(\rho_h(t,y) \,-\, 1)dy\Big]\,\psi(t,x)dtdx.$$

The discrete charge ρ_h converges to $\rho(t,x) = \int_{\mathbb{R}} f(t,x,v)dv$ in $L^\infty(\Omega_T)$ weak-$\star$, where f is solution of the Vlasov equation, thus using the Fubini theorem, we can set $g(t,y) = \int_0^L K(x,y)\,\psi(t,x)dx$ which belongs to $L^1(\Omega_T)$ and satisfies

$$\int_{\Omega_T} \rho_h(t,y)\,g(t,y)dtdy \to \int_{\Omega_T} \rho(t,y)\,g(t,y)dtdy, \quad \text{as } h \to 0.$$

Thus, we have

$$E(t,x) = \int_0^L K(x,y)\,(\rho(t,y)\; - \;1)dy, \qquad \text{and} \qquad \rho(t,y) = \int_{\mathbb{R}} f(t,y,v)dv.$$

Then (f,E) is a solution of the Vlasov-Poisson system.

Regularity of the limit couple (f,E)

The weak formulation infers that the solution of the Vlasov-Poisson system belongs to $C^0([0,T[,\mathcal{D}')$, but observing the electric field E is bounded in $W^{1,\infty}(\Omega_T)$ and the initial data is continuous, the distribution function f is also continuous in (x,v). Let us recall that under our hypothesis, the solution of the Vlasov-Poisson system (1)-(2) is unique, then any subsequence that we considered converges to the same limit, and the sequence $(f_h,E_h)_{h>0}$ converges to the unique solution.

4 Error estimates

Proposition 5 *Under the stability condition (13) on the time step and the condition on the mesh (7), assume the initial data belongs to* $L^1(Q)\cap L^\infty(Q)$ *and is bounded by the function* $R(v)$ *defined previously. Then there exist* $\nu^1_{h,\Delta t}$ *and* $\nu^2_{h,\Delta t}\in\mathcal{M}(Q_T)$ *such that for all* $\varphi\in W^{1,\infty}_c(Q_T)$

$$\int_{Q_T} f_h\Big(\frac{\partial\varphi}{\partial t} + v\,\frac{\partial\varphi}{\partial x}\; + \;E_h(t,x)\,\frac{\partial\varphi}{\partial v}\Big)dtdxdv + \int_Q f_0(x,v)\varphi(0,x,v)dxdv = <\nu^1_{h,\Delta t},\varphi>,$$

and

$$\int_{Q_T} f_h^2\Big(\frac{\partial\varphi}{\partial t} + v\,\frac{\partial\varphi}{\partial x}\; + \;E_h(t,x)\,\frac{\partial\varphi}{\partial v}\Big)dtdxdv + \int_Q f_0^2(x,v)\varphi(0,x,v)dxdv \le <\nu^2_{h,\Delta t},\varphi>$$

The measures satisfy the properties for all $T>0$, $R>0$, *there exists a constant* C *only depending on* T, R, L, f_0, α, ξ, *such that*

$$\begin{aligned}
\nu^1_{h,\Delta t}\Big((0,T)\times(0,L)\times B(0,R)\Big) &\le C\,(\Delta t^{1/2} + h^{1/2} + \|f_0 - f_h(0)\|_{L^1})\\
\nu^2_{h,\Delta t}\Big((0,T)\times(0,L)\times B(0,R)\Big) &\le C\,(\Delta t^{1/2} + h^{1/2} + \|f_0 - f_h(0)\|_{L^2})
\end{aligned}$$

Proof. The idea of the proof is to follow the same argument as for the proof of the convergence of the finite volume scheme to the weak solution of the Vlasov equation given above. We use the Lemma 1 with the convex function $\phi(r)=r$ (resp. $\phi(r)=r^2$) to obtain the bound on the measure $\nu^1_{h,\Delta t}$ (resp. $\nu^2_{h,\Delta t}$). □

From this proposition we obtain the following theorem which gives us an error estimate on the approximation by the finite volume scheme. Now, we will asume the initial data has a compact support.

Theorem 2 *Let $f_0(x,v)$ belong to $W_c^{1,\infty}(Q)$, $\mathcal{M}_h$ be a cartesian mesh of the phase space, let Δt be the time step satisfying the CFL condition, there exits $0<\xi<1$ such that*

$$\frac{\Delta t}{\Delta x_i\,\Delta v_j}\big(\Delta v_j\,|v_j| + \Delta x_i\,|E_i^n|\big) \le 1-\xi, \quad \forall (i,j)\in I\times J, \quad \forall n.$$

If we consider the numerical solution given by the scheme (9), denoted by $f_h(t,x,v)$, and the discrete self-consistent field $E_h(t,x)$ given by (10)

$$\int_{Q_T} e^{-\alpha t}|f(t,x,v)-f_h(t,x,v)|^2 dtdxdv \le C_{1,T}\ (h^{1/2}\ +\ \Delta t^{1/2}) + C_1\|f_0 - f_h(0)\|_2.$$

Proof. As we asume that the initial data has a compact support, for a finite time T there exist $\bar R>0$, such that

$$\forall (t,x)\in (0,T)\times(0,L), \quad supp\Big(f(t,x,.)\Big)\subset B(0,\bar R).$$

Moreover, using the regularity of the initial data, the solution of the Vlasov-Poisson system (E,f) is unique and f belongs to $W^{1,\infty}(Q_T)$. Now, let $\varphi\in W_c^{1,\infty}(Q_T)$, we have

$$\int_{Q_T} f^2\Big(\frac{\partial\varphi}{\partial t} + v\,\frac{\partial\varphi}{\partial x}\ +\ E_h(t,x)\,\frac{\partial\varphi}{\partial v}\Big)dtdxdv + \int_Q f_0^2(x,v)\varphi(0,x,v)dxdv$$
$$= -2\int_{Q_T} f\,(E_h-E)\varphi(t,x,v)\frac{\partial f}{\partial v}dtdxdv.$$

From the Proposition 5, for all $\varphi\in W_c^{1,\infty}(Q_T)$, we observe that $f\,\varphi\in W_c^{1,\infty}(Q_T)$, then using the regularity of the solution

$$-2\int_{Q_T} f_h f\,\Big(\frac{\partial\varphi}{\partial t} + v\,\frac{\partial\varphi}{\partial x}\ +\ E_h(t,x)\,\frac{\partial\varphi}{\partial v}\Big)dtdxdv - 2\int_Q f_0^2(x,v)\varphi(0,x,v)dxdv$$
$$\ge -2 < \nu^1_{h,\Delta t}, f\,\varphi > -2\int_{Q_T} f_h\Big(E-E_h\Big)\varphi(t,x,v)\frac{\partial f}{\partial v}dtdxdv.$$

We finally obtain,

$$\begin{aligned}\int_{Q_T}|f-f_h|^2\Big(\frac{\partial\varphi}{\partial t} + v\,\frac{\partial\varphi}{\partial x}\ +\ E_h(t,x)\,\frac{\partial\varphi}{\partial v}\Big)dtdxdv \ &\ge\ -2\int_{Q_T}(f_h-f)\,(E_h-E)\varphi\,\frac{\partial f}{\partial v}dtdxdv \\ &\quad -2\,|f|_{1,\infty} < \nu^1_{h,\Delta t},\varphi > - < \nu^2_{h,\Delta t},\varphi > .\end{aligned}$$

As we know

$$\begin{aligned}\big|E(t,x)-E_h(t,x)\big| \ &=\ \big|\int_Q K(x,y)[f(t,y,v)-f_h(t,y,v)]dydv\big|,\\ &\le\ \int_Q \big|f(t,y,v)-f_h(t,y,v)\big|dydv.\end{aligned}$$

Thus,

$$2\int_{Q_T}(f_h-f)\,(E_h-E)\varphi\,\frac{\partial f}{\partial v}\,dtdxdv \le 2\,\|\frac{\partial f}{\partial v}\|_\infty\,2\,L\,\bar R\,\int_{Q_T}|\varphi|\,|f-f_h|^2\,dtdxdv.$$

Let us set $\alpha = 4\|\frac{\partial f}{\partial v}\|_\infty L\,\bar R$, and $\omega = max(2\,\bar R; \frac{3}{2}L)$, and consider $k \in C^1(\mathbb{R}^+;[0,1])$ such that

$$k(r) = \begin{cases} 1 & \text{if } r \in [0, \bar R + \omega\, T), \\ \\ 0 & \text{if } r \in [\bar R + \omega\, T + 1, +\infty). \end{cases}$$

and $k'(r) \leq 0, \quad \forall r \in \mathbb{R}^+$. Then, we constuct

$$\varphi(t,x,v) = \begin{cases} k(|(x,v)| + \omega\, t)e^{-\alpha\, t} & \text{if } (t,x,v) \in Q_T, \\ \\ 0 & \text{if } t \geq T. \end{cases}$$

A direct computation gives

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(t,x,v) &= \omega\, k'(|(x,v)| + \omega\, t)\, e^{-\alpha\, t} - \alpha\, k(|(x,v)| + \omega\, t)\, e^{-\alpha\, t}, \\ \frac{\partial \varphi}{\partial x}(t,x,v) &= \frac{x}{|(x,v)|}\, k'(|(x,v)| + \omega\, t)\, e^{-\alpha\, t}, \\ \frac{\partial \varphi}{\partial v}(t,x,v) &= \frac{v}{|(x,v)|}\, k'(|(x,v)| + \omega\, t)\, e^{-\alpha\, t}. \end{aligned}$$

Replacing the derivates by their expression, we finally obtain

$$\begin{aligned} &\int_{Q_T} |f - f_h|^2\, k'(|(x,v)| + \omega\, t)\, e^{-\alpha\, t} \left(w + \frac{v\,x \;+\; E_h(t,x)\, v}{|(x,v)|}\right) dtdxdv \\ &- \alpha \int_{Q_T} |f - f_h|^2\, k(|(x,v)| + \omega\, t)\, e^{-\alpha\, t} dtdxdv \\ &\geq -4\, \|\frac{\partial f}{\partial v}\|_\infty\, L\, \bar R \int_{Q_T} |f - f_h|^2\, k(|(x,v)| + \omega\, t)\, e^{-\alpha\, t}\, dtdxdv \\ &-2\, |f|_{1,\infty}\, \nu^1_{h,\Delta t}\Big((0,T) \times (0,L) \times\, B(0,\bar R)\Big) - \nu^2_{h,\Delta t}\Big((0,T) \times (0,L) \times\, B(0,\bar R)\Big). \end{aligned}$$

But as $k' \;\leq\; 0$ and $\omega = max(2\,\bar R; \frac{3}{2}L)$, we have

$$w + \frac{v\,x \;+\; E_h(t,x)\, v}{|(x,v)|} \;\geq\; 0,$$

and therefore, since $k(|(x,v)| + \omega\, t) = 1$, if $(t,x,v) \in Q_T$

$$\begin{aligned} \int_{Q_T} e^{-\alpha\, t}\; |f - f_h|^2\, dtdxdv \;\;&\leq\;\; C_{1,T} \Big[\nu^1_{h,\Delta t}\Big((0,T) \times (0,L) \times\, B(0,\bar R)\Big) \\ &+\;\; \nu^2_{h,\Delta t}\Big((0,T) \times (0,L) \times\, B(0,\bar R)\Big)\Big]. \end{aligned}$$

From the Proposition 5, we conclude the proof. $\square$

References

[1] J. Batt, The nonlinear Vlasov-Poisson system of partial differential equations in stellar dynamics. *Publications de CNER Mathématiques Pures et Appliquées*, 5: (1983) pp. 1–30.

[2] C. K. Birdsall, A. B. Langdon, Plasma Physics Via Computer Simulation. *McGraw-Hill*, 1985 .

[3] J.P. Boris, D.L. Book, Solution of Continuity Equations by the Method of Flux-Corrected Transport. *J. Comput. Phys.*, 20: (1976) pp. 397–431.

[4] Cheng, G. Knorr, The integration of the Vlasov equation in configuration space. *J. Comput. Phys.*, 22 : (1976) pp. 330–351.

[5] G.-H. Cottet, P.-A. Raviart, Particle Methods for the One Dimensional Vlasov-Poisson Equations. *SIAM J. Numer. Anal.*, 21 -1: (1984) pp. 52–76.

[6] J. Cooper, A. Klimas, Boundary value problems for the Vlasov-Maxwell equation in one dimension. *J. Math. Anal. Appl.*, 75 : (1980) pp. 306–329.

[7] R. J. DiPerna, P. L. Lions, Solutions globales d'équations du type Vlasov-Poisson. *C. R. Acad. Sci.*, Paris Sér.I Math. 307: (1988) pp. 306–329.

[8] E. Fijalkow, A numerical solution to the Vlasov equation. *Comput. Phys. Communications*,116: (1999) pp. 319–328.

[9] R. Eymard, T. Gallouet, R. Herbin, Finite Volume Methods. *Prépublication n^o 97-19 du LATP, UMR 6632 Marseille,*, 1995.

[10] A. J. Klimas, W. M. Farrell A Splitting Algorithm for Vlasov Simulation with Filamaentation Filtration *J. Comput. Phys.*,110: (1994) pp.150–163.

[11] R.J. LeVeque, Numerical Methods for Conservation Laws. *Birkhäuser*, 1992.

[12] P. Mineau, Simulation en Physique des Plasmas. *Thèse de l'Université d'Orléans*, France 1997.

[13] J. Schaeffer, Convergence of a Difference Scheme for the Vlasov-Poisson-Fokker-Planck System in One Dimension. *SIAM J. Numer. Anal.* , 35: (1997) pp. 1149–1175.

[14] M. Shoucri, G. Knorr, Numerical integration of the Vlasov equation. *J. Comput. Phys.*, 14 -1: (1974) pp. 84–92.

[15] E. Sonnendrücker, J. Roche, P. Bertrand, A. Ghizzo, The Semi-Lagrangian Method for the Numerical Resolution of Vlasov Equations. *J. Comput. Phys.* , 149:, (1998) pp.201–220.

[16] Y. Zheng, A. Majda, Existence of Global Weak solutions to One Component Vlasov-Poisson and Fokker-Planck-Poisson Systems in One Space Dimension with Measures as Initial Data. *Comm. Pure Appl. Math.*, 47: (1994) pp. 1365–1401.

Unité de recherche INRIA Lorraine
LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)

Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38330 Montbonnot-St-Martin (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)
Unité de recherche INRIA Sophia Antipolis : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399